Punctures in W-string theory

Suresh Govindarajan[*]

Department of Physics

Indian Institute of Technology

Madras 600 036 INDIA

E-mail: suresh@imsc.ernet.in

ABSTRACT

Using the differential equation approach to W-algebras, we discuss the inclusion of punctures in W-string theory. The key result is the existence of different kinds of punctures in W-strings. We obtain the moduli associated with these punctures and present evidence in existing W-string theories for these punctures. The W_3 case is worked out in detail. It is conjectured that the $(1,3)$ minimal model coupled to two dimensional gravity corresponds to topological W_3-gravity.

1. Introduction

Drinfeld and Sokolov[1] have constructed generalisations of the KdV equation and its related integrable hierarchy. The basic input in their construction is a semi-simple Lie group G and a principal embedding of SL(2) in G. Given this data, they constructed a linear (pseudo-) differential operator from which the generalised KdV hierarchy can be derived. The space of these (pseudo-) differential operators has been shown to have a bi-Hamiltonian structure by Gelfand and Dikii. It was soon realised that classical W-algebras arise precisely from one of these Hamiltonian structures. (See ref. 2 for more details as well as references.) W-string theories can be defined to be those string theories whose underlying chiral algebras are W-algebras.

In a seemingly unrelated development, Hitchin constructed generalisations of the Teichmüller spaces of Riemann surfaces.[3,4] The basic input was similar to that of Drinfeld and Sokolov – a semi-simple Lie algebra G and a principal embedding of SL(2) in G. *Is there a relation between these two constructions?*. This relation was established in

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[*]Talk presented at the Workshop on Frontiers in Field Theory, Quantum Gravity and String Theory, Puri, December 12-21, 1996.

ref. 5 where it was shown that the generalised Teichmüller spaces of Hitchin are indeed the Teichmüller spaces for the W-strings related to the W-algebras obtained from the Drinfeld-Sokolov construction[†]. This connection also provides a Polyakov path integral description for the "W-gravity" sector of a W-string theory.

This however is not the end of the story. The geometric structures (such as complex and projective structures) associated with the generalised Teichmüller spaces of Hitchin are not known unlike the case of the usual Teichmüller space which arises as the space of complex structures on a Riemann surface. In ref. 6 a higher dimensional generalisation of the Riemann surface was constructed to provide geometric insight into the generalised Teichmüller spaces of Hitchin. These manifolds will be referred to as *W-manifolds*. As yet, not much is known about these manifolds. One important property that is however known is that W-diffeomorphisms linearise on W-manifolds. Thus this space plays a role similar to the one played by superspace in the context of supersymmetry. There have been other attempts at constructing geometric structures related to W-gravity. Some recent attempts are refs. 7 and 8.

The results in ref. 6 are abstract since Hitchin's result require the genus $g > 1$. The case of $g = 0, 1$ being excluded by a stability condition required on the Higgs bundle used in Hitchin's construction.[4] To obtain simpler examples, one needs to include the sphere as well as the torus. For these cases, such a stability condition can be satisfied if one includes punctures in the construction. The results which I will describe in this workshop addresses how to describe punctures on W-manifolds and discuss the relationship of these results to W-string theory. Some of the work being reported here has been done in collaboration with T. Jayaraman. A detailed report will appear elsewhere.[9] The main result being reported here is the existence of different kinds of punctures in W-string theory.

The plan of the talk is as follows. The differential equation and the related integrable hierarchy play a central role in the construction of W-manifolds. This is described in sec. 2. In sec. 3, we show how to incorporate punctures in this construction and show the existence of many punctures in W-string theory. For explicitness, we shall use the

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[†]In the Polyakov path integral formulation of the bosonic string, for a given genus g, the path integral over the two dimensional metric reduces to a finite dimensional integral over the moduli space of Riemann surfaces of genus g. This moduli space is obtained from the Teichmüller space by the action of a discrete group called the mapping class group.

example of the W_3-string. We shall see that the W_3-string has two types of punctures. In sec. 4, we provide evidence for these different punctures in existing W-string theories. Finally, we conclude in sec. 5 with some observations.

2.   Constructing W-manifolds

The starting point for the construction of W-manifolds is to consider the linear homogeneous differential equation (possibly obtained from the construction of Drinfeld and Sokolov) associated with the W-algebra of interest. When one considers the group G=SL(n), one obtains the following operator

$$L = d^n + u_2(z)\ d^{n-2} + \cdots + u_n(z) \quad , \tag{2.1}$$

where $d \equiv \frac{d}{dz}$ and the corresponding differential equation

$$L\ f = 0 \quad . \tag{2.2}$$

Here z is a coordinate on some chart on a Riemann surface Σ of genus g. The u_i are assumed to be such that the differential equation is Fuchsian (i.e. the Frobenius method works) and has no regular singular points.

Locally (i.e. on a coordinate chart), the linear differential equation (2.2) has n linearly independent solutions. Let $\{f_i\}\ i = 1, \ldots, n$ be a basis. The Wronskian is defined by

$$W = \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & & f_n' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \quad . \tag{2.3}$$

The non-vanishing of W implies that the basis is linearly independent. The vanishing of the coefficient of $d^{(n-1)}$ in the differential equation (2.2) implies that W is a constant independent of z. The basis can be normalised such that $W = 1$. The basis $\{f_i\}$ can be interpreted as homogeneous coordinates on $\mathbb{CP}^{n-1}$. Thus locally, solving the differential equation provides us with a map from the coordinate chart into $\mathbb{CP}^{n-1}$. (More on the global properties of this map later)

Globally, things are a little more intricate and interesting. First, for the constant condition on the Wronskian to make global sense, we need that W transform like a scalar. This in turn implies that f transforms as a $(1 - n)/2$ differential. Since we are

"solving" the differential equation on a Riemann surface, there exist non-trivial loops. One can analytically continue the basis $\{f_i\}$ along a non-trivial loop γ. Then, the f_i mix amongst each other as follows

$$f_i^{'} = M_i^j(\gamma) \ f_j \quad , \tag{2.4}$$

where M is called the monodromy matrix associated with the loop γ. For the differential equation we are considering, clearly, the matrix $M(\gamma) \in$ G=SL$(n,\mathbb{C})$. The analyticity of f implies that the monodromy matrix can only depend on the homotopy class $\pi_1(\Sigma)$ of the loop γ. By means of analytic continuation of the basis, we can associate to each element of the homotopy class $\pi_1(\Sigma)$, a matrix $M \in G$. This forms the *monodromy group* of the linear differential equation (the group action is the one induced by the composition of loops). The freedom of choice of the basis $\{f_i\}$ implies that this group is defined upto overall conjugation in SL$(n,\mathbb{C})$.

The map from Σ to $\mathbb{CP}^{n-1}$ mentioned earlier globally is a multi-valued (polymorphic) map. This multi-valuedness precisely encodes the monodromy data of the differential equation. A closed loop on Σ will get mapped to a open segment whose endpoints are related by the monodromy matrix (Note that SL$(n,\mathbb{C})$ has a natural action on $\mathbb{CP}^{n-1}$.). Σ can be represented as the quotient of the upper half plane (which is the universal cover of Σ) by a Fuchsian sub-group of SL$(2,\mathbb{R})$ as follows from the uniformisation theorem. Then, the multi-valued map becomes a *single-valued map* from the upper half plane to $\mathbb{CP}^{n-1}$ with the image being co-dimension $(n-2)$ surface.

This image can be extended to obtain a $(n-1)$ dimensional image by augmenting the differential equation by means of $(n-2)$ times of the generalised KdV hierarchy. The times are defined by the equations

$$\frac{d}{dt_p}L = [(L^{p/n})_+, L] \quad , \tag{2.5}$$

where $p = 2, \ldots, (n-1)$ and by $(L^{p/n})_+$, we mean the differential operator part of the pseudo-differential operator $(L^{p/n})$. Thus the u_i's which entered the differential equation are now functions of the KdV times. The importance of the times of the generalised KdV hierarchy is that these furnish *isomonodromic deformations* of the differential equation, i.e., they do not alter the monodromy group of the differential equation. This extended image on being quotiented by the monodromy group provides a higher dimensional generalisation of Riemann surfaces. The moduli space of complex structures is related to

the generalised Teichmüller spaces of Hitchin. We refer to these objects as *W-manifolds*. The non-linear W-diffeomorphisms turn out be a sub-group of linear diffeomorphisms on W-manifolds. See refs. 10,6 for more details. This construction is abstract and as mentioned earlier, not much is known about these manifolds. Thus it is of interest to obtain things in a more explicit fashion. One way to do this is to include the cases of genus 0 and 1 into the construction of W-manifolds. However, in order to obtain stable configurations (in the sense of Higgs bundles in Hitchin's construction) one needs to include punctures. This is addressed in the next section.

3. Describing Punctures

The differential equation approach provides a rather simple way of including punctures into the construction of W-manifolds. Punctures are defined as *regular singular points with specified monodromy conjugacy class*. In the case of Riemann surfaces, the monodromy associated with a puncture is parabolic, i.e., it is conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and is an element of $\mathrm{SL}(2,\mathbb{R})$. This imposes the following condition on $u_2(z)^{12}$

$$u_2(z) = \frac{1}{4z^2} + \cdots \quad , \tag{3.1}$$

where $z = 0$ is a puncture. The monodromy matrix around the puncture is described by two real parameters and this is the moduli associated with the puncture.

We shall now consider of the third order differential equation which is relevant for the W_3 case. Again, we shall define a puncture to be a regular singular point and specified monodromy conjugacy class. What are the allowed monodromies? Using the second order case as a guide, we shall consider unipotent matrices – there is more than one of them. We find that there are two distinct possibilities:

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \tag{3.2}$$

with the requirement that the monodromy matrix is in $\mathrm{SL}(3,\mathbb{R})$. The moduli associated with these punctures are obtained by the following procedure. A typical monodromy matrix of type P is of the form APA^{-1} for some $A \in \mathrm{SL}(3,\mathbb{R})/\mathcal{I}$, where $\mathcal{I}$ is the subgroup of $SL(3,\mathbb{R})$ matrices which commute with P. One can check that the dimension of $\mathcal{I}$ is 2 and hence the number of (real) parameters associated with a P type puncture is

$8 - 2 = 6$. Similarly, the number of (real) parameters associated with a Q-type puncture is 4.

Thus, the differential equation approach to W-geometry suggests that there exists two types of punctures with (complex) moduli three and two respectively. The moduli associated with a genus g surface with N_P P-type punctures and N_Q Q-type punctures is given by

$$(8g - 8 + 3N_P + 2N_Q) \quad . \tag{3.3}$$

It is of interest to look for rigid objects, i.e., those without any moduli. These occur at genus zero for $N_P = 2$, $N_Q = 1$ (this configuration will be represented by the symbol $\langle PPQ \rangle$) and for $N_P = 0$, $N_Q = 4$ (this configuration will be represented by the symbol $\langle QQQQ \rangle$). The choice of these symbols is to emphasise the similarity to a topological model to be discussed in the next section.

For the case of W_n-strings, there are $(n-1)$ types of punctures with (complex) moduli ranging from $(n^2 - n)/2$ to $(n-1)$.

4. Evidence for the existence of different kinds of punctures

The results of the previous section suggest the existence of two kinds of punctures in W_3 string theory. Is this a reasonable suggestion? Different kinds of punctures have already occurred in superstring theory – the NS and R types. Thus this is not a radical suggestion. We shall provide direct evidence for this from existing W_3-strings! In addition, we shall also show how a certain existing topological model is the perfect candidate for topological W_3-gravity.

4.1. W_3 string theory

The section makes extensive use of results of P. West and his collaborators. Please see ref. 13,14 for more details and related references. We shall now consider the W_3-string associated with pure W_3-gravity. There is considerable evidence that this theory is closely related to the Ising model coupled to 2d gravity.[15] The fields in this theory are:

- *Gravity sector:* the spin-2 ghost system (b, c) and the spin-3 ghost system (d, e).

- *Liouville sector:* a scalar ϕ

- *Matter sector:* Scalars x^μ (for pure W_3 gravity, there is one scalar field)

The states are labeled by the primaries of the Ising model – $1, \sigma, \epsilon$ and are

$$V(1,0) \;=\; c\partial e\; e\; e^{i\beta(1;0)\phi}\; V^x(1) \quad , \tag{4.1}$$

$$V(\sigma,0) \;=\; c\partial e\; e\; e^{i\beta(\sigma;0)\phi}\; V^x(\frac{15}{16}) \quad , \tag{4.2}$$

$$V(\epsilon,0) \;=\; (c\, e - \frac{i}{522}\partial e\; e)\; e^{i\beta(\epsilon;0)\phi}\; V^x(\frac{1}{2}) \tag{4.3}$$

where by $V^x(h)$, we mean a vertex operator in the matter sector with dimension h; $\beta(1;n) = (8-8n)iQ/7$, $\beta(\sigma;n) = (7-4n)iQ/7$, $\beta(\epsilon;n) = (4-8n)iQ/7$ and $Q^2 = 49/8$. The first two states given above occur at "standard" ghost number three while the last one (labeled by ϵ) occurs at ghost number two. This unusual occurance was first observed by S. K. Rama.[16]

In the operator formulation of string theory, the number of moduli are given by the number of antighost insertions required to obtain a non-vanishing amplitude. Here there are two types of anti-ghosts b and d given by the spin two and three systems respectively. Thus, the ghost number of a state provides a direct count of moduli associated with a state. We thus obtain that states labeled by 1 and σ have three moduli per insertion (and thus correspond to P-type punctures) and states labeled by ϵ have two moduli per insertion (and thus correspond to Q-type punctures).

This simple minded counting can be seen to be correct by studying scattering amplitudes in the theory.[17] One can look for states at non-standard ghost numbers and this leads to an infinite set of states labeled by

$$V(1,n) \quad ; \quad V(\sigma,n) \quad ; \quad V(\epsilon,n) \tag{4.4}$$

Are these new states? As we shall see now, these cannot be considered as new states. Define the following operators

$$\widehat{S} \;\equiv\; \oint \{d - \frac{5i}{3\sqrt{58}}\partial b + \cdots\}e^{i\beta^s\phi} \quad , \tag{4.5}$$

$$\widehat{P} \;\equiv\; [a\cdot x + \phi, Q_{BRST}] \quad , \tag{4.6}$$

where $\beta^s = -2iQ/7$ and $[\widehat{S}, Q_{BRST}] = 0$. One can show that $V(\sigma,n) = (\widehat{S}^2\widehat{P})^n V(\sigma,0)$ with similar relations holding for the other operators too. The form of $\widehat{S}$ shows that the insertion of a $\widehat{S}$ operator is equivalent to carrying out a $\int d$ insertion. By similar arguments, one can show that the operators given in eqn. (4.4) are not new states.

Thus one can choose to work with $n = 0$ states and put in as many insertions of $\widehat{S}$ and $\widehat{P}$ as are necessary. Consider a scattering amplitude involving N_1, N_σ and N_ϵ states of type 1, σ and ϵ respectively. The ghost number and Liouville background charge conditions give us the numbers N_S and N_P, of $\widehat{S}$ and $\widehat{P}$ insertions. The number of extra moduli (in comparison with the bosonic case) required are given by the expression $(N_S - N_P)$ which using the expressions given by West work out to $(2N_1 + 2N_\sigma + N_\epsilon - 5)$. Adding the bosonic moduli, we get that the total moduli counting works out to be $(3N_1 + 3N_\sigma + 2N_\epsilon - 8)$. Discounting the -8 in the expression (this is due to zero modes of the ghosts on the sphere), we see that there are three moduli per puncture for 1 and σ operators and two moduli per puncture for the ϵ operator[‡]. Comparing the moduli count with that in eqn. (3.3), we obtain that the 1 and σ correspond to punctures of type P and ϵ corresponds to punctures of type Q.

4.2. A conjecture

It is of interest to construct a topological model for W_3-gravity. Such a model would probe the moduli spaces associated with W_3-gravity. Rather than directly construct this model, we shall show that an existing topological theory[§] – the $(1,3)$ minimal model coupled to two dimensional gravity. This model is closely related to the Gaussian point of the two-matrix model. This has been studied in some detail by Dijkgraaf and Witten.[19]

We shall provide some relevant details of this model. The reader is referred to the reference cited earlier for more details. The *small phase space* of this model consists of two operators labeled P and Q, in the notation of Witten and Dijkgraaf. At the topological point, the only non-vanishing correlators at genus 0 are $\langle PPQ \rangle$ and $\langle QQQQ \rangle$. In topological theories, ghost charge is related to the number of moduli. Further, at the topological point, the only non-vanishing objects are those which saturate ghost charge which implies that they have no moduli associated with them. This is exactly what we had seen earlier with regard to the punctures of type P and Q.

In continuum Liouville theory, one has vertex operator representations of these

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[‡]P. West interprets $(N_S - N_P + N_\epsilon)$ as the number of extra moduli. Our interpretation removes this distinction between the ϵ operator and the other operators.

[§]W. Lerche has pointed out that there is evidence to suggest that certain W-minimal models couples to W-gravity give rise to topological W-matter coupled to topological W-gravity.[18] We thank him for the information.

operators. They are given by[20]

$$P = e^{-3\phi_L/2\sqrt{3}} \, e^{iX/2\sqrt{3}} \quad , \tag{4.7}$$

$$Q = e^{-2\phi_L/2\sqrt{3}} \quad , \tag{4.8}$$

with the Liouville background charge $Q_L = -8/2\sqrt{3}$. The Liouville charge conservation condition is

$$\sum_i p_L^i = Q_L$$

where p_L^i are the Liouville charges of the $i-th$ vertex operator in a correlation function. Clearly, $\langle PPQ \rangle$ and $\langle QQQQ \rangle$ saturate this condition and the Liouville charges of the P and Q operators are in the ratio $3:2$ as required from our moduli argument.

This shows that this model has all the right properties to be a theory of topological W_3-gravity. It has two operators in the small phase space which implies that it has two types of punctures. Further, the moduli counting also works out right. Finally, this model exhibits multicritical behaviour and the first critical point corresponds to the Ising model coupled to gravity which we saw is related to W_3 gravity.[19,20] This might suggest the following conjecture: the $(1, n)$ model coupled to two dimensional gravity should represent topological W_n gravity. However, from $n = 4$ onwards, the rigid objects in this model are different from the one obtained from the differential equation approach pursued in this paper! Thus for $n > 3$, the $(1, n)$ model coupled to two dimensional gravity does not correspond to topological W_n-gravity.

The topological theory probes the moduli spaces of W_3 gravity. Is there something known about these spaces? Hitchin's result has to be extended to include punctures. This extension has been carried out for the case of P-type punctures.[11] However, so far the Higgs bundle corresponding to Q-type punctures have not been constructed.

5. Conclusion and Outlook

In this talk, we have seen the existence of many different kinds of punctures in W-string theory. The genus zero case can be studied in greater detail and new insight may possibly be obtained. For example, the new moduli which appear in W-strings may be understood better. In the bosonic string, the four punctured sphere has one modulus which is described by means of a cross-ratio. Thus all four point functions depend on the positions of the punctures in terms of this cross-ratio. In the W_3-case, the first non-trivial modulus appears at genus zero for $N_P = 0$ and

$N_Q = 6$. This must be described by means of a generalised cross-ratio. Goncharov has constructed a generalised cross-ratio[21] in proving certain conjectures of Zagier connecting trilogarithms to zeta functions. This cross-ratio seems to have the right properties (such as transformations under $SL(3)$), and hence is the candidate for the new W-modulus. This also suggests an interesting interplay between Algebraic Number Theory and correlation functions in W_3 conformal field theory.

Some open problems with regard to the W-manifolds include the construction of Higgs bundles which encompass all types of punctures, more global information about these manifolds like the metric etc. Since the W-manifolds seem to be higher dimensional generalisation of Riemann surfaces which are the worldsheets for strings, it is natural to think of these as the worldsheets for p-branes. If this were true, then these manifolds will provide a topological expansion analogous to the genus expansion for strings.

One of the big problems with working with W-symmetries, is their non-linearity. The linearisation provided by the W-manifolds should provide some simplification in realising W-symmetries and hence might lead to the construction of many more systems realising this symmetry.

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